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ON BANACH SPACES WITH UNCONDITIONAL BASES

BY

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ABSTRACT

Let X be a Banach space with a sequence of linear, bounded finite rank operators $R_n\colon X \to X$ such that $R_n R_m = R_{\min(n,m)}$ if $n \neq m$ and $\lim_{n\to\infty} R_n x = x$ for all $x \in X$. We prove that, if $R_n - R_{n-1}$ factors uniformly through some l_p and satisfies a certain additional symmetry condition, then X has an unconditional basis. As an application we study conditions on $\Lambda \subset \mathbb{Z}$ such that $L_\Lambda =$ closed span $\{z^k : k \in \Lambda\} \subset L_1(\mathbb{T})$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, has an unconditional basis. Examples include the Hardy space $H_1 = L_{\mathbb{Z}_+}$.

1. Introduction

Let X be a given separable Banach space (real or complex). We study an abstract condition on X which ensures that X has an unconditional basis without constructing an explicit one. Then we apply our results to spaces of the form

$$L_\Lambda = \text{closed span}\{z^k : k \in \Lambda\} \subset L_1(\mathbb{T})$$

for special subsets $\Lambda \subset \mathbb{Z}$ where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. In particular we want to find out what abstract condition on $H_1 = L_{\mathbb{Z}_+}$ is responsible for the existence of an unconditional basis in H_1.

Fix some p with $1 \leq p \leq \infty$. We say that a sequence of linear operators $U_n\colon X \to X$ factors uniformly through l_p if there are linear operators $T_n\colon X \to l_p$ and $S_n\colon l_p \to X$ with $U_n = S_n T_n$ and $\sup_n \|T_n\| \cdot \|S_n\| < \infty$.

It is clear that we can replace l_p by any $\mathcal{L}_p$-space or by $l_p^{m_n}$, for some m_n, in the preceding condition.

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If, in addition, all operators U_n are projections then it is easily seen that U_nX is uniformly isomorphic to $(T_nS_n)^2l_p$ and $(T_nS_n)^2\colon l_p \to l_p$ is a projection.

If $U_n - U_{n-1}$, instead of U_n, factors uniformly through l_p then so does $U_{n+k} - U_n$ for any fixed k. (This follows from $U_{n+k} - U_n = \sum_{j=1}^{k}(U_{n+j} - U_{n+j-1})$.)

A sequence of bounded linear operators $R_n\colon X \to X$ of finite rank is called a commuting approximating sequence (c.a.s.) if $\lim_{n\to\infty} R_nx = x$ for all $x \in X$ and $R_nR_m = R_{\min(n,m)}$ whenever $n \neq m$. If there exists such a sequence then X is said to have the commuting bounded approximation property (CBAP).

If there is a c.a.s. $\{R_n\}_{n=1}^{\infty}$ consisting of projections, i.e. where in addition $R_nR_n = R_n$ for all n, then X is said to have a finite dimensional Schauder decomposition (FDD). It is well-known that there are Banach spaces with CBAP which do not have FDD ([10], see also [12]).

On the other hand, if X has a c.a.s. $\{R_n\}_{n=1}^{\infty}$ such that the operators $R_n - R_{n-1}$ factor uniformly through some l_p then X has a basis, i.e. a special c.a.s. $\{P_n\}_{n=1}^{\infty}$ consisting of projections where, in addition, dim $(P_n - P_{n-1})X = 1$ for all n ([8]). (In the following always put $R_0 = R_{-1} = \cdots = 0$.)

Our aim is to derive a similar result for unconditional bases. Recall that X is said to have an unconditional basis if it has a c.a.s. $\{P_n\}_{n=1}^{\infty}$ consisting of projections such that $\dim(P_n - P_{n-1})X = 1$ for all n and

$$\sup_{\theta_n\in\{\pm1\}} \|\sum_n \theta_n(P_n - P_{n-1})x\| < \infty \quad \text{for all } x \in X.$$

THEOREM I: *Let X have a c.a.s. $\{R_n\}_{n=1}^{\infty}$ satisfying the following:*

(1.1) *$R_n - R_{n-1}$ factors uniformly through l_p for some $p \in [1,\infty]$,*

(1.2) *there is $\lambda > 0$ such that, for any linear $U_n\colon X \to X$ with $\|U_n\| \leq 1$ and any sequence of indices $\{k_n\}_{n=1}^{\infty}$ with $k_n \neq k_{n'}$ if $n \neq n'$, we have*

$$\left\|\sum_n (R_{k_n} - R_{k_n-1})U_n(R_n - R_{n-1})x\right\| \leq \lambda\|x\| \quad \textit{for all } x \in X.$$

Then X has an unconditional basis.

We postpone the proof of Theorem I to sections 2 and 3. Here we discuss some applications of Theorem I (with $p = 1$) centered around the classical Hardy space $H_1 = L_{\mathbb{Z}_+}$.

Consider a sequence of integers n_k and real numbers $a > 0$, $b > 0$ such that

$$(1.3) \qquad n_0 = 0 < n_1 < n_2 < \cdots \quad \text{and} \quad an_k \leq n_{k+1} - n_k \leq bn_k \quad \text{for all } k.$$

(The prototype for such a sequence is $n_k = 2^k$.) Then we prove in section 4

THEOREM II: *Consider* $\Lambda \subset \mathbb{Z}_+$ *such that there are* $q, m_{k,j}, r_{k,j} \in \mathbb{Z}_+$, $j = 1, \dots, q$, $k = 1, 2, \dots$, *with*

$$\Lambda \cap [n_k, n_{k+2}] = \bigcup_{j=1}^{q} (m_{k,j}\mathbb{Z} + r_{k,j}) \cap [n_k, n_{k+2}] \quad \text{for all } k. \tag{1.4}$$

Then L_Λ *has an unconditional basis.*

Remarks: Note that two consecutive intervals $[n_k, n_{k+2}]$ and $[n_{k+1}, n_{k+3}]$ always overlap. It is easy to construct non-trivial examples of Λ satisfying (1.4). If we take $q = m_{k,j} = 1$ and $r_{k,j} = 0$ then we obtain the well-known

COROLLARY ([9, 2, 13]): H_1 *has an unconditional basis.*

2. Proof of Theorem I if $\sup_n \dim(R_n - R_{n-1})X = \infty$

We start with

2.1. LEMMA: *Let* X *have a c.a.s.* $\{R_n\}_{n=1}^{\infty}$. *Then (1.2) is equivalent to the following condition.*

(2.1) *There is* $\lambda > 0$ *such that, for any sequence of indices* k_n *with* $k_n \neq k_{n'}$ *if* $n \neq n'$ *and any linear* $U_n \colon X \to (R_{k_n} - R_{k_n - 1})X$ *with* $\|U_n\| \leq 1$, *we have*

$$\left\| \sum_n U_n (R_n - R_{n-1}) x \right\| \leq \lambda \|x\|, \quad x \in X.$$

Proof: Of course, by definition, (2.1) implies (1.2). Now assume (1.2) and consider $U_n \colon X \to (R_{k_n} - R_{k_n - 1})X$. Put

$$Ux = \sum_n (R_{k_n} - R_{k_n - 1}) U_n (R_n - R_{n-1}) x,$$

$$Vx = \sum_n (R_{k_n - 1} - R_{k_n - 2}) U_n (R_n - R_{n-1}) x \quad \text{and}$$

$$Wx = \sum_n (R_{k_n + 1} - R_{k_n}) U_n (R_n - R_{n-1}) x, x \in X.$$

Then (1.2) implies $\|U\|, \|V\|, \|W\| \leq \lambda$. We have

$$\sum_n U_n (R_n - R_{n-1}) x = (U + V + W) x, \quad x \in X.$$

This proves (2.1). ∎

Now we consider a c.a.s. $\{R_n\}_{n=1}^{\infty}$ of X such that $\sup_n \dim(R_n - R_{n-1})X = \infty$ and there are linear $T_n\colon X \to l_p$, $S_n\colon l_p \to X$ for some $p \in [1,\infty]$ with $S_nT_n = R_n - R_{n-1}$ and $\sup_n \|T_n\| \cdot \|S_n\| < \infty$. We always assume

$$\text{either} \quad p = 2 \quad \text{or} \quad \sup_n d(T_nX, l_2^{\dim T_nX}) = \infty \tag{2.2}$$

where $d(\cdot,\cdot)$ is the Banach–Mazur distance. This is no restriction. Indeed, if $\sup_n d(T_nX, l_2^{\dim T_nX}) < \infty$ then we just replace p by 2.

Moreover, we assume (1.2). Notice that (1.2) remains true if we replace $\{R_n\}_{n=1}^{\infty}$ by an arbitrary subsequence $\{R_{n_k}\}_{k=1}^{\infty}$.

At first we mention

2.2. LEMMA: *There is a subsequence $\{R_{n_k}\}_{k=1}^{\infty}$ of $\{R_n\}_{n=1}^{\infty}$ satisfying the following:*

(i) *$R_{n_k} - R_{n_{k-1}}$ factors uniformly through l_p, too.*

(ii) *For each positive integer n there is an index k, a subspace $E_n \subset (R_{n_k} - R_{n_{k-1}})X$ and a projection $Q_n\colon X \to E_n$ such that*

$$\sup_n d(E_n, l_p^n) < \infty, \quad \sup_n \|Q_n\| < \infty$$

and

$$Q_nR_{n_j} = R_{n_j}Q_n = \begin{cases} Q_n, & j \ge k, \\ 0, & j < k. \end{cases}$$

Proof: [8], Lemma 2.3. ∎

In the following we assume without loss of generality that $R_{n_j} = R_j$ for all j. Then, in particular, for every n there is an index k_n such that $l_p^n \cong E_n \subset (R_{k_n} - R_{k_n-1})X$ and

$$Q_nR_j = R_jQ_n = \begin{cases} Q_n, & j \ge k_n, \\ 0, & j < k_n. \end{cases} \tag{2.3}$$

Put

$$Y = \text{closed span } \{(R_1x, (R_2 - R_1)x, (R_3 - R_2)x, \ldots) : x \in \cup_{n=1}^{\infty} \operatorname{Fix} R_n\}$$

regarded as subspace of $(\sum_n \oplus (R_n - R_{n-1})X)_{(p)}$. Then we easily obtain, using (2.2),

2.3. LEMMA: *Y is isomorphic to l_p, if $1 \le p < \infty$, and to c_0, if $p = \infty$.*

Proof: [8], Lemma 2.1. ∎

From now on we take $T_n: X \to Y$ with

$$T_n x = ((R_k - R_{k-1})(R_{n+1} - R_{n-2})x)_{k=1}^{\infty}$$

and $S_n: Y \to X$ with

$$S_n((R_k - R_{k-1})x)_{k=1}^{\infty} = (R_n - R_{n-1})x.$$

We easily obtain $S_n T_n = R_n - R_{n-1}$. (Recall that $(R_k - R_{k-1})(R_{n+1} - R_{n-2}) = 0$, if $k < n-2$ and $k > n+2$, and $(R_n - R_{n-1})(R_{n+1} - R_{n-2}) = R_n - R_{n-1}$.) We have

$$\begin{aligned} &T_n R_m = 0, \quad R_m S_n = 0, \quad \text{if } m < n-2, \\ \text{and} \quad &T_n R_m = T_n, \quad R_m S_n = S_n, \quad \text{if } m > n+1. \end{aligned} \tag{2.4}$$

Moreover, we define $\tilde{R}_n: Y \to Y$ by

$$\tilde{R}_n((R_k - R_{k-1})x)_{k=1}^{\infty} = ((R_k - R_{k-1})R_n x)_{k=1}^{\infty}.$$

Then $\{\tilde{R}_n\}_{n=1}^{\infty}$ is a c.a.s. of Y with

$$S_n \tilde{R}_n = R_n S_n, \quad T_n R_n = \tilde{R}_n T_n \quad \text{and} \quad (id - \tilde{R}_n) T_n S_n = \tilde{R}_n (id - \tilde{R}_n). \tag{2.5}$$

We fix a subspace $F_n \subset Y$ with

$$\sup_n d(F_n, l_p^{\dim F_n}) < \infty, \quad T_n X \subset F_n \quad \text{and} \quad \tilde{R}_n Y \subset F_n. \tag{2.6}$$

Conclusion of the proof of Theorem I if $\sup_n \dim(R_n - R_{n-1})X = \infty$*:* Our strategy is to split X into $X = Y_1 \oplus \sum_n \oplus E_n$, where Y_1 and the spaces E_n are invariant for the operators R_n. This allows us to alter R_n and to produce FDD-projections P_n which still satisfy (1.1) and (1.2). After another modification the summands $(P_n - P_{n-1})X$ are uniformly isomorphic to $l_p^{m_n}$-spaces which, together with (1.2), yield an unconditional basis. Parallel to R_n on X we study $\tilde{R}_n$ on $Y \sim l_p$ which behaves "locally like X".

We assume (1.1), (1.2), (2.4)–(2.6). Moreover, using Lemma 2.2, for each n we find an index k_n, a subspace $E_n \subset (R_{k_n} - R_{k_n - 1})X$ and a projection $Q_n: X \to E_n$ such that $k_1 < k_2 < \cdots$, $\sup_n \|Q_n\| < \infty$, $\sup_n d(E_n, F_n) < \infty$ and (2.3) is satisfied. Let $I_n: E_n \to F_n$ be isomorphisms with $\sup_n \|I_n\| \cdot \|I_n^{-1}\| < \infty$. Put

$$Qx = \sum_n (R_{k_n} - R_{k_n - 1}) Q_n (R_{k_n} - R_{k_n - 1}) x = \sum_n Q_n x, \tag{2.7}$$

$x \in X$. According to (2.3), in view of (1.2), Q is a bounded projection from X onto $Y_2 :=$ closed span$(\bigcup_{n=1}^{\infty} E_n)$. Put $Y_1 = (id - Q)X$. Hence $X = Y_1 \oplus Y_2$. (2.3) implies

$$R_m Q = Q R_m \quad \text{and} \quad Q R_m (id - R_m) = 0 \quad \text{for all } m \tag{2.8}$$

and the operators $R_n|_{Y_2}$ are the FDD-projections of $Y_2 = \sum_m \oplus E_m$ (where some R_n, for different n, might coincide on Y_2). In particular, Y_1 and Y_2 are invariant under all R_m. Moreover, (2.3) and (2.7) also yield $R_n(id - R_n)Q = QR_n(id - R_n) = 0$. Using the definitions of T_n and S_n and (2.8) we obtain

$$T_n(id - R_n)(id - Q)S_n = \tilde{R}_n(id - \tilde{R}_n). \tag{2.9}$$

We have $X = Y_1 \oplus \sum_m \oplus E_m$. Define

$$P_m \colon Y_1 \oplus \sum_m \oplus E_m \to Y_1 \oplus \sum_m \oplus E_m$$

by

$$\begin{aligned} P_n(y, (e_k)) = (&R_n y + (id - Q)S_n I_n e_n, (e_1, \ldots, e_{n-1}, \\ &I_n^{-1} T_n (id - R_n) y + I_n^{-1}(id - \tilde{R}_n) I_n e_n, 0, 0, \ldots)). \end{aligned} \tag{2.10}$$

(2.8) shows that the definition makes sense. We easily infer

$$P_m P_n = P_{\min(m,n)} \quad \text{if } |n - m| \geq 3$$

(see (2.3) and (2.4)).

As a consequence of (2.5), (2.8) and (2.9) we also obtain $P_n^2 = P_n$. (Recall that $(id - Q)y = y$ if $y \in Y_1$.) (2.10) implies

$$\begin{aligned} &(P_n - P_{n-1})(y, (e_k)) = \\ &\quad ((R_n - R_{n-1})y + (id - Q)(S_n I_n e_n - S_{n-1} I_{n-1} e_{n-1}), (\underbrace{0, \ldots, 0}_{n-2}, \\ &\quad e_{n-1} - I_{n-1}^{-1}(id - \tilde{R}_{n-1}) I_{n-1} e_{n-1} - I_{n-1}^{-1} T_{n-1}(id - R_{n-1}) y, \\ &\quad I_n^{-1} T_n (id - R_n) y + I_n^{-1}(id - \tilde{R}_n) I_n e_n, 0, 0, \ldots)). \end{aligned} \tag{2.11}$$

Next we claim that $P_n - P_{n-1}$ factors uniformly through l_p. To this end put $\hat{Y} = F_{n-1} \oplus F_n \oplus F_{n-1} \oplus F_n$. Then $d(\hat{Y}, l_p^m)$, where $m = \dim \hat{Y}$, does not depend on n. Define $\hat{T}_n \colon X = Y_1 \oplus \sum_m \oplus E_m \to \hat{Y}$ by

$$\hat{T}_n(y, (e_k)) = (T_{n-1} y, T_n y, I_{n-1} e_{n-1}, I_n e_n)$$

and $\hat{S}_n\colon \hat{Y} \to X = Y_1 \oplus \sum_m \oplus E_m$ by

$$\begin{aligned}\hat{S}_n(a,b,c,d) =&((id-Q)S_n b + (id-Q)(S_n d - S_{n-1}c), \underbrace{(0,\dots,0}_{n-2},\\ &I_{n-1}^{-1}c - I_{n-1}^{-1}(id-\tilde{R}_{n-1})c - I_{n-1}^{-1}(id-\tilde{R}_{n-1})a,\\ &I_n^{-1}(id-\tilde{R}_n)b + I_n^{-1}(id-\tilde{R}_n)d, 0, 0, \dots)).\end{aligned}$$

Then, in view of (2.11), we have $\hat{S}_n\hat{T}_n = P_n - P_{n-1}$ and $\sup_n \|\hat{T}_n\| \cdot \|\hat{S}_n\| < \infty$. Thus, $P_n - P_{n-1}$ and hence also $P_n - P_{n-3}$ factors uniformly through l_p. Since the operators $P_{3n} - P_{3n-3}$ are projections, the spaces $(P_{3n} - P_{3n-3})X$ are uniformly isomorphic to uniformly complemented subspaces of l_p. (2.11) implies, together with (2.3), (2.4), (2.7) and (2.8), that

$$\begin{aligned}P_n - P_{n-1} = &(id-Q)(R_n - R_{n-1}) + (id-Q)S_n I_n Q_n (R_{k_n} - R_{k_n-1})\\ &-(id-Q)S_{n-1}I_{n-1}Q_{n-1}(R_{k_{n-1}} - R_{k_{n-1}-1}) + Q_{n-1}(R_{k_{n-1}} - R_{k_{n-1}-1})\\ &-I_{n-1}^{-1}(id-\tilde{R}_{n-1})I_{n-1}Q_{n-1}(R_{k_{n-1}} - R_{k_{n-1}-1})\\ &-I_{n-1}^{-1}T_{n-1}(id-R_{n-1})Q(R_{n+1}-R_{n-3}) + I_n^{-1}T_n(id-R_n)Q(R_{n+2}-R_{n-2})\\ &+I_n^{-1}(id-\tilde{R}_n)I_nQ_n(R_{k_n} - R_{k_n-1}).\end{aligned}$$

The definitions of S_n, T_n, I_n, Q_n, E_n and (2.8) show that all preceding eight summands map into spaces of the form $(R_m - R_{m-1})X$ for some m. This proves that (2.1) remains true if we replace R_n by P_n. Using Lemma 2.1, we see that (1.2) is also true for P_n and hence for P_{3n} instead of R_n. In particular, the spaces $A_n := (P_{3n} - P_{3n-3})X$ are the summands of an unconditional FDD of X. (Here take in (1.2) the operators with $U_k x = \theta_k x$, $x \in X$, for arbitrarily fixed $\theta_k \in \{\pm 1\}$.)

Finally, we cut the summands A_n into two subsummands and merge them into new summands. We proceed as follows. Recall, all A_n are uniformly complemented in l_p. We split the spaces A_k into uniformly complemented subspaces $A_k = B_k \oplus C_k$ and define indices $0 = j_0 < j_1 < j_2 < \cdots$ such that $A_k = C_k$ and $B_k = \{0\}$ if $k \notin \{j_1, j_2, \dots\}$.

We use induction to define the special indices j_n. Start with $j_0 = 0$. Then assume that we have already $j_0 < j_1 < \cdots < j_{n-1}$ with $A_{j_k} = B_{j_k} \oplus C_{j_k}$, $k = 1, \dots, n-1$. Consider A_n. A_n is uniformly complemented in an l_p^M-space G_n. There are two cases.

Case $n \notin \{j_1, \dots, j_{n-1}\}$. Using Lemma 2.2 with P_{3n} instead of R_n we find $j_n > \max(j_{n-1}, n)$ such that A_{j_n} contains a uniformly complemented copy of G_n. Hence we can split $A_{j_n} = B_{j_n} \oplus C_{j_n}$ such that, with $D_n = A_n \oplus B_{j_n}$, the Banach–Mazur distance $d(D_n, l_p^m)$, where $m = \dim D_n$, does not depend on n.

Case $n \in \{j_1, \dots, j_{n-1}\}$. Here we have already $A_n = B_n \oplus C_n$. Find similarly $j_n > j_{n-1}$ and a decomposition $A_{j_n} = B_{j_n} \oplus C_{j_n}$ such that, for $D_n = C_n \oplus B_{j_n}$, the Banach–Mazur distance $d(D_n, l_p^k)$, with $k = \dim D_n$, does not depend on n.

Using this procedure we find an FDD, $X = \sum_n \oplus D_n$, with $\sup_n d(D_n, l_p^{q_n}) < \infty$ for $q_n = \dim D_n$ where all D_n are uniformly complemented subspaces of $A_n \oplus A_{j_n}$. Recall, (P_{3n}) (in place of (R_n)) satisfies (1.2) and we have $A_n = (P_{3n} - P_{3n-3})X$. Since $P_{3n} - P_{3n-3}$ are projections, we conclude that there is a constant $\tau > 0$ satisfying the following.

For every choice of indices l_n with $l_n \neq l_{n'}$ if $n \neq n'$ and every linear U_n: $D_n \to D_{l_n}$ with $\|U_n\| \leq 1$ we have $\|U\| \leq \tau$, where $U \sum_k d_k = \sum_k U_k d_k$.

Now, let $\{e_{i,n}\}_{i=1}^{q_n}$ be the unit vector basis of the $l_p^{q_n}$-spaces D_n. Fix $\theta_{i,n} \in \{\pm 1\}$ arbitrarily and put $U_n \sum_{i=1}^{q_n} \alpha_{i,n} e_{i,n} = \sum_{i=1}^{q_n} \alpha_{i,n} \theta_{i,n} e_{i,n}$ for all coefficients $\alpha_{i,n}$. Then we obtain $\|U\| \leq \tau \sup_n \|U_n\|$ for the operator U: $X \to X$ with $U e_{i,n} = \theta_{i,n} e_{i,n}$ for all i and n. Hence $\{e_{i,n}\}_{i=1,n=1}^{q_n,\infty}$ is an unconditional basis of X. ∎

3. Proof of Theorem I if $\sup_n \dim(R_n - R_{n-1})X < \infty$

Find uniformly bounded projections Q_n: $X \to R_n(id - R_n)X$. Put

$$P_n = R_n + Q_n(R_{n+1} - R_n).$$

Then we have $P_n P_m = P_{\min(n,m)}$ whenever $|n - m| \geq 2$ or $n = m$. (Notice that $Q_n(R_{n+1} - R_n)R_n = Q_n(R_n - R_n^2) = R_n - R_n^2$, $R_{n+1}Q_n = Q_n$, $Q_n R_n Q_n = R_n Q_n$ and $R_m Q_n = 0$ if $m < n$.) Hence $\{P_{2n}\}_{n=1}^{\infty}$ are FDD-projections. We obtain

$$P_n - P_{n-2} = (R_n - R_{n-1}) + (R_{n-1} - R_{n-2}) + Q_n(R_{n+1} - R_n) - Q_{n-2}(R_{n-1} - R_{n-2}).$$

This implies that (2.1) and hence, in view of Lemma 2.1, (1.2) remains true if we replace R_n by P_{2n}. Moreover, we have $N := \sup_n \dim(P_n - P_{n-2})X < \infty$. Find bases $\{x_{i,n}\}_{i=1}^{k_n}$ of $(P_{2n} - P_{2n-2})X$, where $k_n = \dim(P_{2n} - P_{2n-2})X \leq N$, with uniformly bounded basis constants. As before, fix arbitrary $\theta_{i,n} \in \{\pm 1\}$. Then, in view of (1.2), the operators U: $X \to X$ with $U x_{i,n} = \theta_{i,n} x_{i,n}$ for all i and n are uniformly bounded. Hence $\{x_{i,n}\}_{i=1,n=1}^{k_n,\infty}$ is an unconditional basis of X. ∎

4. Proof of Theorem II

Here, let $\|\cdot\|_1$ be the norm in $L_1(\mathbb{T})$ and $\|\cdot\|_\infty$ the norm in $L_\infty(\mathbb{T})$. Consider a sequence of indices n_k satisfying (1.3). Put

$$R_k\Big(\sum_j \alpha_j z^j\Big) = \sum_{|j|\le n_k} \alpha_j z^j + \sum_{n_k<|j|\le n_{k+1}} \frac{n_{k+1}-|j|}{n_{k+1}-n_k}\alpha_j z^j.$$

Then R_k is well-defined on $L_1(\mathbb{T})$ and on $L_\infty(\mathbb{T})$. It is well-known (see e.g. [8]) that

$$\|R_k\| \le \frac{n_{k+1}+n_k}{n_{k+1}-n_k}$$

in either norm. Hence, in view of (1.3), $\|R_k\| \le 1 + 2/a$. This means the operators R_k are uniformly bounded on $L_1(\mathbb{T})$ as well as on $L_\infty(\mathbb{T})$. Moreover, for any $\Lambda \subset \mathbb{Z}$, L_Λ is invariant under the operators R_k. We clearly obtain that $\{R_k\}_{k=1}^\infty$ is a c.a.s. on all L_Λ-spaces.

Put $R(\sum_j \alpha_j z^j) = \sum_{j\ge 0} \alpha_j z^j$. R is well-defined on all trigonometric polynomials. [8] Lemma 4.1. and (1.3) imply

4.1. LEMMA: *The operators $R(R_n - R_{n-1})$ are uniformly bounded on $L_\infty(\mathbb{T})$ as well as on $L_1(\mathbb{T})$.*

Now we concentrate on $H_1 = L_{\mathbb{Z}_+}$. We make use of

4.2. THEOREM (Stein, [11, 4]): *Let μ_n be complex numbers satisfying*

$$\sup_n \max(|\mu_n|, n|\mu_n - \mu_{n-1}|) < \infty.$$

Then the operator $V\colon H_1 \to H_1$ with

$$V(\sum_{k\ge 0} \alpha_k z^k) = \sum_{k\ge 0} \alpha_k \mu_k z^k$$

is bounded.

With (1.3) the preceding theorem implies that, for any choice of $\theta_k \in \{\pm 1\}$, the operator V_θ with $V_\theta f = \sum_k \theta_k (R_k - R_{k-1}) f$, $f \in H_1$, is bounded. (Split V_θ into two convolution operators where the numbers μ_j have the form

$$\theta_k \frac{n_{k+1}-j}{n_{k+1}-n_k} \quad \text{if } n_k \le j \le n_{k+1},$$
$$\theta_k \frac{j-n_{k-1}}{n_k-n_{k-1}} \quad \text{if } n_{k-1} \le j \le n_k,$$

and use (1.3).) With the uniform boundedness principle and the Khintchine inequality we obtain

4.3. COROLLARY: *There are constants* $c_1 > 0$ *and* $c_2 > 0$ *such that every* $f \in H_1$ *satisfies*

$$c_1\|f\|_1 \le \|\sqrt{\sum_k |(R_k - R_{k-1})f|^2}\|_1 \le c_2\|f\|_1.$$

If $m, n \in \mathbb{Z}_+$, f and g are trigonometric polynomials of degree m and n, resp., and $\epsilon > 0$, then we easily find $N = N(m, n, \epsilon) \in \mathbb{Z}_+$ such that

$$(1-\epsilon)\|f + z^N g\|_1 \le \frac{1}{2}\|f + z^N g\|_1 + \frac{1}{2}\|f - z^N g\|_1 \le (1+\epsilon)\|f + z^N g\|_1$$

and

$$(1-\epsilon)\|f + z^N g\|_\infty \le \|f\|_\infty + \|g\|_\infty \le (1+\epsilon)\|f + z^N g\|_\infty.$$

Using the Khintchine inequality and induction we obtain

4.4. LEMMA: *Let* $k_n \in \mathbb{Z}_+$, $n = 1, 2, \ldots$, *be arbitrary. Then there are universal constants* $c_1 > 0$ *and* $c_2 > 0$ *such that, whenever* $m_n \in \mathbb{Z}_+$ *are large enough and* f_n *are trigonometric polynomials of degree* $\le k_n$, *we have*

$$c_1\|\sum_n z^{m_n} f_n\|_1 \le \|\sqrt{\sum_n |f_n|^2}\|_1 \le c_2\|\sum_n z^{m_n} f_n\|_1$$

and

$$c_1\|\sum_n z^{m_n} f_n\|_\infty \le \sum_n \|f_n\|_\infty \le c_2\|\sum_n z^{m_n} f_n\|_\infty.$$

Now we prove

4.5. LEMMA: *There is a universal constant* $\mu > 0$ *satisfying the following.*

Whenever k_n *are indices with* $k_n \ne k_{n'}$ *for* $n \ne n'$ *then there are* $m_1 < m_2 < \cdots$ *such that*

$$\left\|\sum_n (R_{k_n} - R_{k_n-1})\bar{z}^{m_n} f\right\|_1 \le \mu\|f\|_1 \quad \text{for all } f \in H_1.$$

Proof: (1.3) implies that $\sup_j(n_{j+1} - n_j) = \infty$. Hence, for each j, we find $p_j \in \mathbb{Z}_+$ and $m_j \in \mathbb{Z}_+$ such that

$$n_{p_j} < m_j + n_{k_j-2} < m_j + n_{k_j+2} < n_{p_j+1}.$$

The definition of the R_k implies

$$(4.1) \qquad (R_{k_j} - R_{k_j-1})\bar{z}^{m_j}(R_{p_j+1} - R_{p_j-1}) = (R_{k_j} - R_{k_j-1})\bar{z}^{m_j}.$$

Let $r_j\colon [0,2\pi] \to \{\pm 1\}$ be the jth Rademacher function. Put

$$U_\theta f = \sum_l r_l(\theta)\bar{z}^{m_l}(R_{p_l+1} - R_{p_l-1})f$$

and

$$V_\theta f = \sum_j r_j(\theta)(R_{k_j} - R_{k_j-1})f,$$

f a polynomial. Then, in view of Theorem 4.2 and the uniform boundedness principle, there is a constant $\tau > 0$ independent of p_j, k_j and θ such that $\|V_\theta\| \le \tau$.

Furthermore, put $Wf = \sum_j (R_{k_j} - R_{k_j-1})\bar{z}^{m_j} f$. Then we obtain, using (4.1), the Khintchine inequality and Corollary 4.3,

$$\begin{aligned}\|Wf\|_1 &= \| \int V_\theta U_\theta f d\theta \|_1 \\ &\le \int \|V_\theta U_\theta f\|_1 d\theta \\ &\le \tau \int \|U_\theta f\|_1 d\theta \\ &\le \tau \| \sqrt{\sum_l |(R_{p_l+1} - R_{p_l-1})f|^2} \|_1 \\ &\le \mu \|f\|_1\end{aligned}$$

for some universal constant μ. ∎

4.6. Lemma: *The R_n, regarded as operators on H_1, satisfy (1.2).*

Proof: Let $c_1 > 0$ and $c_2 > 0$ be the constants of Lemma 4.4. Fix indices k_n with $k_n \neq k_{n'}$ if $n \neq n'$ and fix $m_n \in \mathbb{Z}_+$ so large that

$$c_1 \| \sum_n z^{m_n} f_n \|_\infty \le \sum_n \|f_n\|_\infty \le c_2 \| \sum_n z^{m_n} f_n \|_\infty \tag{4.2}$$

for all trigonometric polynomials $f_n \in (R_{k_n} - R_{k_n-1})L_1(\mathbb{T})$ and that the assertion of Lemma 4.5 holds.

Fix linear $U_n\colon H_1 \to H_1$ with $\|U_n\| \le 1$. Define $V_n\colon L_1(\mathbb{T}) \to L_1(\mathbb{T})$ and $Q_n\colon L_1(\mathbb{T}) \to L_1(\mathbb{T})$ by

$$V_n f = z^{m_n} U_n (R_n - R_{n-1}) R f \quad \text{and} \quad Q_n f = (R_{k_n} - R_{k_n-1})\bar{z}^{m_n} f.$$

Hence $Q_nV_n = (R_{k_n} - R_{k_n-1})U_n(R_n - R_{n-1})R$. Clearly, the operators V_n and Q_n are uniformly bounded by Lemma 4.1. Now, we consider

$$Q_n^* = z^{m_n}(R_{k_n} - R_{k_n-1}) \colon L_\infty(\mathbb{T}) \to C(\mathbb{T})$$

and V_n^*. (4.2) implies $\|\sum_n V_n^* Q_n^* f\|_\infty \le c_2 \|\sum_n Q_n^* f\|_\infty$ for any $f \in L_\infty(\mathbb{T})$.

Define $\tilde{W}$: closed span$(\bigcup_n Q_n^* L_\infty(\mathbb{T})) \to L_\infty(\mathbb{T})$ by $\tilde{W}\sum_n f_n = \sum_n V_n^* f_n$ for $f_n \in Q_n^* L_\infty(\mathbb{T})$. Then we obtain $\|\tilde{W}\| \le c_2$. Using (4.2) and the (L_∞-valued) Hahn–Banach theorem we can extend $\tilde{W}$ to a linear operator $W\colon L_\infty(\mathbb{T}) \to L_\infty(\mathbb{T})$ with $\|W\| \le c_2$. By definition we have $WQ_n^* = V_n^* Q_n^*$. Regard $L_1(\mathbb{T})$ as subspace of $L_1^{**}(\mathbb{T})$. Then we obtain $Q_n W^*|_{L_1(\mathbb{T})} = Q_nV_n$. Lemma 4.5 implies $\|\sum_n Q_n W^* f\|_1 \le c_2\mu \|f\|_1$ for any $f \in H_1$. Put $\lambda = c_2\mu$. Finally, we have, for $f \in H_1$,

$$\begin{aligned}\sum_n Q_n W^* f &= \sum_n Q_n V_n f \\ &= \sum_n (R_{k_n} - R_{k_n-1})U_n(R_n - R_{n-1})f. \qquad \blacksquare\end{aligned}$$

Conclusion of the proof of Theorem II: We take the preceding operators R_k restricted to $X = L_\Lambda$. Put $\Lambda_k = \bigcup_{j=1}^q (m_{k,j}\mathbb{Z} + r_{k,j})$ and let $P_k\colon L_1(\mathbb{T}) \to L_{\Lambda_k}$ be the projection with

$$P_k(\sum_{j\in\mathbb{Z}} \alpha_j z^j) = \sum_{j\in\Lambda_k} \alpha_j z^j.$$

These projections are uniformly bounded ([8], Lemma 4.2).

Now, define $T_n\colon X \to L_1(\mathbb{T})$ by $T_n = id_X$ and $S_n\colon L_1(\mathbb{T}) \to X$ by $S_n = P_nR(R_n - R_{n-1})$, which makes sense according to (1.4). Hence S_n and T_n are uniformly bounded (see Lemma 4.1) and we have $S_nT_n = (R_n - R_{n-1})|_X$. This proves (1.1).

Finally, let $U_n\colon X \to X$ be linear operators with $\|U_n\| \le 1$ and fix indices k_n such that $k_n \ne k_{n'}$ if $n \ne n'$. Then $U_n \circ P_n|_{H_1}$ is an operator from H_1 into H_1. We have

$$(R_k - R_{k-1})X \subset \text{closed span}\{z^j : j \in [n_{k-1}, n_{k+1}] \cap \Lambda\}.$$

This implies

$$\sum_n (R_{k_n} - R_{k_n-1})U_n(R_n - R_{n-1})x = \sum_n (R_{k_n} - R_{k_n-1})U_nP_n(R_n - R_{n-1})x$$

for all $x \in X$. Now, Lemma 4.6 shows that (1.2) holds for X and $R_n|X$. Thus Theorem I proves Theorem II. $\blacksquare$

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